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Margulis' construction of a family of expander graphs, i.e. sparse graphs with strong connectivity properties

Recall that a graph consists of a set of nodes, V, and a set of edges, E, between some pairs of nodes. In a complete graph, there is an edge between any pair of nodes, thus being far from sparse. In a connected graph any two nodes are connected by a sequence of edges. Expander graphs are sparse graphs with strong connectivity properties. Expander graphs have found a lot of applications in complexity theory, design of robust computer networks, and the theory of error-correcting codes, to mention some.

A way of describing a graph G is by considering the adjacency matrix. The adjacency matrix $A = (a_{ij})$ is a $n \times n$ square matrix where the number of columns and rows equals the number of nodes ($n = |G|$), and such that

$$a_{ij} = \begin{cases} 1 & \text{if } \{i,j\} \in E \\ 0 & \text{if } \{i,j\} \notin E \end{cases}$$

We can also define the adjacency operator;
$A : \ell^2(V) \to \ell^2(V)$ on functions $f : G \to \mathbb{C}$ by the formula

$$Af(v) = \sum_{\{v,w\} \in E} f(w)$$

i.e. we take the sum of the function over all the neighbours of v. The number of neighbours of a node $v \in V$ is called the valence of v, denoted $\mathrm{val}(v)$.
The adjacency matrix is symmetric with eigenvalues

$$\lambda_1 \geq \cdots \geq \lambda_n$$

The eigenvalues reveals different properties of the graph. For a k-regular graph, i.e. a graph for which every node has precisely k neighbours, we have $\lambda_1 = k$ and $\lambda_n \geq -k$. We also have $\lambda_n = -k$ if and only if G contains a non-empty bipartite graph as a connected component. For a k-regular graph on n vertices, we have

$$\sum_{i=1}^{n} \lambda_i = 0 \quad \text{and} \quad \sum_{i=1}^{n} \lambda_i^2 = nk$$

For any subset W of nodes in G we define ∂W as the set of nodes, outside W but with an edge into W, i.e.

$$\partial W = \{v \in V \setminus W \mid \exists w \in W \text{ such that} \{v,w\} \in E\}$$

For any set W of nodes in G we use the notation

$$\varepsilon(W, \partial W) = \{\{v,w\} \in E \mid w \in W,\, v \in \partial W\}$$

Definition. Let $G = (V, E)$ be a finite graph. The expansion constant (or Cheeger constant) $h(G)$ of G is given by

$$h(G) = \min_{W}\{\frac{|\varepsilon(W, \partial W)|}{|W|}\}$$

where W is any non-trivial subset of nodes i G, of cardinality $|W| \leq \frac{1}{2}|V|$.

Definition. A graph is a (d, ϵ)-expander if it is d-regular and $h(G) \geq \epsilon$.

Notice that $\varepsilon(W, \partial W) \leq d|W|$. In other words, $h(G)$ is the smallest possible ratio between the number of edges exiting from W and the size of W, when W is a set of vertices that is non-empty, but not too big. The expansion constant gives valueable information about the connectedness of G. In fact, one can show that G is connected if and only if $h(G) > 0$. Another fact is that if W is a subset of nodes with

$$\frac{|W|}{|V|} = \delta < \tfrac{1}{2}$$

you have to remove at least $\delta \cdot h(G) \cdot |V|$ to disconnect W from the rest of the graph.

Definition. A family $\{ G_i \}_{i \in \mathbb{N}}$ of finite non-empty connected graphs $G_i = (V_i, E_i)$ is an expander family if there exist constants $\nu \geq 1$ and $h > 0$, not depending on i such that:

i) The number of nodes tends to ∞, i.e.

$$|G_i| \to \infty \quad \text{as} \quad i \to \infty$$

ii) The number of neighbours is limited (sparsity), i.e. for each $i \in \mathbb{N}$ and $v \in V_i$, we have

$$\text{val}(v) \leq \nu$$

iii) Connectedness is controlled, i.e. for each $i \in \mathbb{N}$, the expansion constant satisfies

$$h(G_i) \geq h > 0$$

Even if it is rather easy to find examples of expander graphs, it was not until 1973 known how to construct a family of such graphs. Margulis came up with the following construction:
For every n, let G_n be a graph with vertex set $\mathbb{Z}_n \times \mathbb{Z}_n$, i.e. G_n has n^2 nodes. Define four functions,

$$\begin{aligned} S(a,b) &= (a, a+b) \\ T(a,b) &= (a+b, b) \\ s(a,b) &= (a+1, b) \\ t(a,b) &= (a, b+1) \end{aligned}$$

where addition takes place modulo n.
A vertex (a, b) in G_n is connected to the 8 other vertices, given by

$$s, s^{-1}, t, t^{-1}, S, S^{-1}, T, T^{-1}$$

and Margulis showed that $h(G_n) \geq 0.46$ for all n. It follows that $\{G_n\}$ is an expander family of constant valence 8, and expansion constant bounded below by 0.46.

Very soon after this first construction in 1973, other examples were constructed.:

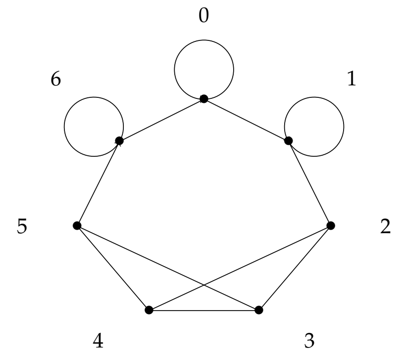

Figure 1: The graph of the last example for $p = 7$. Notice that a loop counts as one edge.

Let p be a prime and let $V = \mathbb{Z}_p$. We define a 3-regular graph $G = (V, E)$ where the edges are of two types, $(x, x+1)$ and (x, x^{-1}) for each $x \in \mathbb{Z}_p$. (Put $0^{-1} = 0$). This is a $(3, \epsilon)$-expander for some fixed $\epsilon > 0$ and any prime p.