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Asymptotic properties of groups

A popular family of mathematical objects is the so-called Lie groups, named after the Norwegian mathematician Sophus Lie (1842–1899). Lie groups are objects that describes the symmetries of geometrical objects, such as rotational symmetry in three dimensional space. Sophus Lie was inspired by earlier work of Abel and Galois on solutions of algebraic equations. Abel's proof of the non-solvability of the fifth degree equation in radicals, and Galois' ground-breaking theory for connecting solutions of polynomial equations to certain automorphism groups of field extensions, are both brilliant examples of how one can understand the details by extending the horizon. Lie's idea was to introduce a similar way of studying symmetries of differential equations. It has ever since been an important task to understand the structure of such groups, in order to approach the solution of the underlying differential equations.

An abstract group is a set with a binary operation, satisfying certain properties, as associativity and existence of inverses. The binary operation can be addition, multiplication or composition of elements or functions, depending on which group we have in mind. Groups can have finitely or infintely many elements, and they might have a rather complex structure. But groups can also have a more or less trivial structure, such as the additive group $\{0\}$ of only one element. Other examples of groups are the integers $\mathbb{Z}$ with ordinary addition as the binary operation, the set of invertible $n \times n$-matrices with multiplication as the binary operation, or the set of symmetries of a cube under composition, which in fact is the same group as the set of permutations of the set $\{A, B, C, D\}$.

In the same way as the symmetry group of a cube acts on the vertices of the cube, an abstract group can act on an arbitrary set. The action must reflect the binary structure of the group, i.e. if $\rho_g(x)$ denotes the action of the group element g on the element $x \in X$, it is neccessary that $\rho_{gg'}(x) = \rho_g(\rho_{g'}(x))$.

The atoms in the world of groups are the simple groups. For finite groups, simple groups are completely classified, due to work of a lot of mathematicians through many decades, among those the 2008 Abel Prize Laureate John Griggs Thompson played a leading role.

Groups may have additional structure, compatible with the binary operation of the group, such as topological groups. A topological group is a group which is also a topological space with a continuous binary operation.
As an example of a topological group, we consider the circle group

$$S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$$

The group operation is given by adding angles. In fact, the circle group is what we call a compact topological group.

An important property of a group is the size of the group. For finite groups we can count the elements to find the order of the group, but for infinite groups it is not that simple. We need more sophisticated measures, and we focus on two different properties, amenability and the Kazhdan property (T).

In order to define amenability we use the notion of a Følner sequence. A Følner sequence for an action of a group G on a countable set X is a sequence of finite sets which "fills up" X and such that the action on X "don't move too much." The precise definition is as follows;

Definition. Let G be a group acting on a countable set X. A Følner sequence for the action is a sequence $F_1 \subset F_2 \subset \dots$ of finite subsets of X such that $\bigcup_{j=1}^{\infty} F_j = X$ and such that

$$\lim_{j \to \infty} \frac{|gF_j \,\Delta\, F_j|}{|F_j|} = 0$$

for all $g \in G$ and where Δ denotes the symmetric difference, i.e. $A \,\Delta\, B = (A \cup B) \setminus (A \cap B)$.
A discret countable group G is amenable if it contains at least one Følner sequence for the action of the group on itself.

An example of an amenable group is the integers $\mathbb{Z}$. It has a Følner sequence $F_j = [-j, \dots, j]$ with union all of $\mathbb{Z}$ and where

$$(z + F_j) \,\Delta\, F_j = [j+1, \dots, j+z] \cup [-j, \dots, -j+z]$$

has cardinality $2z$. Since z is fixed, the limit of the fraction by $|F_j| = 2j + 1$ is 0.
An equivalent definition for countable discrete groups, due to J. Dixmier, is that there are unit vectors ξ in $\ell^2(G)$ such that $||g\xi - \xi||$ tends to 0 for each $g \in G$. We say that ξ is an almost invariant vector. Notice that finite, solvable and finitely generated groups of polynomial growth are amenable.
The growth rate of a group is a well-defined notion from asymptotic analysis. To say that a finitely generated group has polynomial growth means the number of elements of length (relative to a symmetric generating set) at most n is bounded above by a polynomial function $P(n)$. The order of growth is then the least degree of any such polynomial function P.

The other property we consider is the so-called Kazhdan property (T).

Definition. Let G be a locally compact group and $\rho : G \to U(H)$ a unitary representation of G on a Hilbert space H. For any $\epsilon > 0$ and a compact subset $S \subset G$ a unit vector $\xi \in H$ is called an (ϵ, S)-invariant if

$$||\rho(g)\xi - \xi|| < \epsilon \quad \forall g \in S$$

We say that G has Kazhdan property (T) if every unitary representation of G that has an (ϵ, S)-invariant unit vector for any $\epsilon > 0$ and any compact subset S, has a non-zero invariant vector.

Using the Dixmier definition of amenability we notice the relation between the two definitions; Amenability is equivalent to the existence of a (ϵ, S)-invariant unit vector $\xi \in H$, and Kazhdans property (T) says that if G that has an (ϵ, S)-invariant unit vector for any $\epsilon > 0$ and any compact subset S, then it has a non-zero invariant vector. We have some examples of groups with property (T): Finite groups, compact topological groups and simple real Lie groups of real rank at least two, including the special linear groups $SL_n(\mathbb{R})$ for $n \geq 3$.
We also look at some examples of groups that do not have property (T): The additive groups of integers $\mathbb{Z}$, or of real numbers $\mathbb{R}$, noncompact solvable groups, nontrivial free groups and free abelian groups and the special linear groups $SL_2(\mathbb{Z})$ and $SL_2(\mathbb{R})$.
To illustrate some of the technicalities in this univers we give a proof of the fact that any finite group has property (T):

Proof. Let G be a finite group. Let $\rho : G \to U(H)$ be a unitary representation of G that has a (ϵ, S)-invariant unit vector ξ for any subset $S \subset G$ and $\epsilon > 0$. We can obviously assume that

$$\sup_{s \in G} ||\rho(s)\xi - \xi|| < \sqrt{2}$$

Then it follows that

$$\begin{aligned} 1 - &\frac{||\rho(s)\xi - \xi||^2}{2} \\ &= 1 - \tfrac{1}{2}\langle \rho(s)\xi - \xi, \rho(s)\xi - \xi \rangle \\ &= 1 - \tfrac{1}{2}(\langle \rho(s)\xi, \rho(s)\xi \rangle + \langle -\xi, \rho(s)\xi \rangle \\ &\quad + \langle \rho(s)\xi, -\xi \rangle + \langle -\xi, -\xi \rangle) \\ &= \tfrac{1}{2}(\langle \xi, \rho(s)\xi \rangle + \langle \rho(s)\xi, \xi \rangle) \\ &= \Re\langle \rho(s)\xi, \xi \rangle > 0 \end{aligned}$$

Let

$$\eta = \sum_{g \in G} \rho(g)\xi$$

By construction this vector is invariant. It is also non-zero, since

$$\Re\langle \eta, \xi \rangle \neq 0$$

by the above argument. Thus we have showed that almost invariant vectors produce a non-zero invariant vector. □

We also give a proof of a case where we have the opposite conclusion; The group $\mathbb{R}$ does not satisfy Kazhdans property (T).

Proof. Let $\lambda : \mathbb{R} \to \ell^2(\mathbb{R})$ be the left regular representation of $\mathbb{R}$, i.e.

$$\lambda(t)f(x) = f(x - t)$$

Let $Q \subset \mathbb{R}$ be a compact subset, bounded by M, and let $\epsilon > 0$ be a real number. Consider the interval $I = [a, b]$ in $\mathbb{R}$ such that $b - a > \frac{2M}{\epsilon^2}$, and let

$$\xi = (b-a)^{-\frac{1}{2}} \chi : \mathbb{R} \to \mathbb{R}$$

where χ is the characteristic function of $[a, b]$. Then we have

$$||\xi||^2 = \int_{\mathbb{R}} \xi^2 \, dx = \int_{\mathbb{R}} \frac{\chi^2}{b-a} \, dx = 1$$

and ξ is unitary. Furthermore, we have

$$\begin{aligned} (\lambda(t)\xi - \xi)^2 &= (\xi(x-t) - \xi(x))^2 \\ &= \frac{1}{b-a}(\chi_{[a,a+t]} + \chi_{[b,b+t]}) \end{aligned}$$

where we have used that $|t| < b - a$. Therefore

$$\begin{aligned} ||\lambda(t)\xi - \xi||^2 &= \int_{\mathbb{R}} (\lambda(t)\xi - \xi)^2 \, dx \\ &= \frac{2|t|}{b-a} < \frac{2M}{b-a} < \epsilon^2 \end{aligned}$$

It follows that the regular representation almost has invariant vectors.
On the other hand, the regular representation has no invariant vectors. In fact, suppose

$$\lambda(t) f(x) = f(x-t) = f(x)$$

for all $t \in \mathbb{R}$. Then f is constant, different from 0. But non-negative constant functions do not belong to $\ell^2(\mathbb{R})$. $\square$