L SYLOW

ON ABEL'S WORKS AND PLANS IN HIS LAST YEARS, ILLUSTRATED BY DOCUMENTS THAT CAME TO LIGHT AFTER THE SECOND EDITION OF HIS WORKS

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BY

L SYLOW

Your Majesty!
Ladies and Gentlemen!

Abel again! That is what you may have exclaimed on seeing the subject of the Congress's first meeting – has the subject of Abel not been exhausted yet here in Norway! Yes, we have published his works twice, we have celebrated the 100th anniversary of his birth, and his name was not forgotten at the University's 100th anniversary either. That is the situation.

But for Norwegian mathematicians, Abel is still of greater importance than he is for others. He was the first Norwegian to make a contribution of lasting significance to scientific development. No doubt there was an even earlier brilliant Norwegian-born mathematician, but Caspar Wessel's astonishing work has been lying unheeded for almost a hundred years, despite having been printed in the publications of the Royal Danish Academy of Sciences and Letters. It was only in 1894 that Prof. Thiele drew attention to the fundamental significance of this, but by then the rediscovery had taken place long ago. Things were different with Abel. *His* fame is that he has been an example and a stimulus for all Norwegian mathematicians. I might be tempted to say that everything of any significance that has been generated in pure mathematics, here in this country, stems directly or indirectly from Abel. The subjects he studied have had a captivating attraction, and his exhaustive treatment of them and the emphasis he placed on rigour also acted as an example. In addition, the fundamental nature of his discoveries has fired the ambition of young mathematicians. Personally, I am able clearly to trace his influence on all of those, who before those here present have occupied a professorship at our university – even his old teacher Holmboe. His admittedly rather heavy textbooks in elementary mathematics strove unmistakably for precision, which has certainly had an influence on the older generations. In this I see, at least partially, an influence of the pupil on the teacher. The definition that Holmboe gave in his lectures for the differential ratio is remarkable to me. He says: When a function's growth is developed in power series according to the increment in the absolute variable, that term in the expansion, which contains the first power, is called the differential of the function. This restriction to analytical functions is very reminiscent of Abel's critique – I sense conversations between Holmboe and Abel behind this attentiveness. Also on Holmboe's successors, Broch and Bjerknes, even on the geometer Lie, as different as the latter was from Abel with regard to mathematical of interests and working method, Abel's influence is unmistakable. – I beg you to forgive this digression.

This is the background: new details have been added to Abel's mathematical history since the second edition of Abel's Collected Works. This meeting's outstanding Swedish member, Prof. Mittag-Leffler, whose admiration for Abel and great interest in his memory are known to all mathematicians, has deservedly called attention to two unknown manuscripts for Abel from collections abroad. The first of these is a whole treatise, which Abel sent to his friend Crelle, the publisher of the "Journal für die reine und angewandte Mathematik"; the second is a letter, also to Crelle. Prof Mittag-Leffler has had these manuscripts printed in the special volumes of Acta Mathematica[1] on the occasion of the Abel Centennial Anniversary, with the letter as a photographic facsimile. He also did me the honour of forwarding in advance a copy to me. However, due to lack of time, it was not possible for me to include them in my presentation of Abel's scientific progress in our University's Festschrift in 1902. Since, as far as I know, Prof. Mittag-Leffler has not published anything further about them other than his concise indication of their content, with which he accompanied them in the Acta, I felt I ought to make them the subject of further examination.

For the results of this, I would like to ask for a few moments' attention.

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[1] Volume 26 (1902), volume 27 (1903) and volume 28 (1904) of Acta Mathematica were all dedicated to the memory of Abel, and contained articles by the most famous mathematicians of the day (among them Poincaré and Hilbert), taking Abel's mathematical discoveries as inspiration and starting point.

I will begin with the treatise. It is a link in the well-known competition, in part even the struggle for precedence, between Abel and the other major author in the field of elliptic functions, Jacobi. I must therefore provide a reminder about the way this took place, and because of the way in which the activities of both men interacted, I cannot avoid discussing the various works relating to this and the time of their publications.

Abel's first work on elliptic functions was entitled "Recherches sur les fonctions elliptiques" and was printed in the second and third volumes of Crelle's Journal, of which the first was published on September 20, 1827, and the second on May 26, 1828. However the last section of the treatise had been received by Crelle long before, as early as February 12. The treatment of the transformation of the elliptic functions is contained in the last article. Since, however, *also in September 1827*, Jacobi had reported a theorem about the same subject, but without proof, in Astronomische Nachrichten, Abel added a postscript to his treatise, in which he shows that Jacobi's discovery is only a special case of the theorem that he had himself just proven in complete generality, only later. The new treatise, which was printed in Acta's Special Volumé in 1902, has the same title as the first: Recherches sur les fonctions elliptiques, but with the addition *Second Mémoire,* and is dated August 27, 1828. During the period between February 12 and August 27, Abel had also written another substantial and celebrated treatise, "Solution d'un problème gènéral concernant la transformation des fonctions elliptiques". In 1827, and therefore before the last part of Abel's work was available, Jacobi had already presented a proof of his theorem, based on the same understanding of the nature of elliptic functions as Abel had himself put forward. Abel wanted to show how much further progress he had made than Jacobi, in exactly that theory in which Jacobi had gained an advantage as far as time of publication was concerned. He triumphantly achieved his goal – it was precisely this treatise that provoked the well-known exclamation by Jacobi in a letter to the French mathematician, Legendre: "Elle est au-dessus de mes éloges comme elle est au-dessus de mes propres travaux". I would like to interject immediately that there was no lack of warm recognition of Jacobi's merits on Abel's part. Highly appreciative statements were made both in treatises and particularly in a letter to the same Legendre.

This treatise "Solution d'un problème etc" as well as a later, perhaps even more famous work on the same subject, was printed in the same periodical as Jacobi's manuscript had appeared – Astronomische Nachrichten. That was in June and November, 1828.

But now, in a short note in Crelle's Journal, Jacobi again reported an elegant solution to Abel's multiplication equation – that is, a solution to the problem of division of the elliptic function's argument. In the first part of his oldest treatise, Abel had also given a solution to the equation, which is important because it appears to have led him to his Transformation Theory. But it cannot be denied that, from an algebraic perspective, Jacobi's solution has more merits compared to Abel's. Jacobi's Note appeared on March 25, 1828, just at the time when the second part of Abel's "Recherches" was being published by Crelle.

Then it was Abel's turn. He wanted to show that this was nothing new to him either. He wanted to take up the division problem again to show that it led to other properties, unknown until then, of the elliptic functions, and the new treatise "Second Mémoire" was destined for this purpose. Under these circumstances, however, it must evoke astonishment and regret that only the first section was published in Crelle's Journal, and then under a different title. It seems particularly unfortunate that the second section was not included at all, because that section in particular contains an exhaustive treatment of Jacobi's formula, which is analogous with Abel's treatment of similar algebraic questions.

The question arises immediately – does this mean that Crelle did not *want* to print everything. It appears also as if Prof. Mittag-Leffler has perceived the omission of the other sections as an unfortunate error by Crelle. For my part, I find it difficult to believe that any change was made to the treatise without Abel's approval. The relationship between Crelle and Abel was such that I cannot conceive of any unfairness on Crelle's part. I emphatically want to say this since I have also heard from other mathematicians and in other connections – critical words about Crelle's relationship with Abel, which do not seem to me to be fair. In addition, I believe that I can show that there is something in the treatise, and particularly with that second section, that may have made Abel himself decide to cut it out.

However, I must first discuss another peculiarity of this "Second Mémoire". Among the papers Abel left behind there are two fragments that were printed in the second edition of Abel's Collected Works, of which the first is the beginning of a treatise with exactly the same title as that printed in the Acta, and which appears to contain a complete first section, while the second fragment is undoubtedly part of the same treatise. With regard to paper, format and handwriting, both have a great similarity to the letter which is reproduced as a photograph in the 27th volume of Acta Mathematica.[2] It appears as though all of the works, which Abel sent from here [Kristiania] to Crelle, were written on that kind of letter paper and with that kind of close, fine handwriting. Therefore, Abel had also been working on a *second* treatise with the same title, and he completed a good part of the final editing. And these two treatises are quite different. The one printed in Acta, deals in its first two sections with the division equation; in the third section, Abel's addition theorem is stated specifically for elliptic functions. The fourth contains connections between the roots of the period division equation, which was deduced from what was proved in §2. The fifth states 3 theorems, of which one provides what is now called the transformation equation's monodromy group, with an indication of the proof.

The last treatise, on the other hand, that from which the fragments were taken in Abel's Collected Works, Second Edition, only deals with the transformation equation. It gives the monodromy group with full proof and also the relations between the roots of the period division equation, which can be deduced from the solution to the equation. These relations, which are also mentioned, but without proof, in §4 of the first treatise, are presented simpler here than in the first treatise and may replace the earlier ones altogether.

If one goes through the treatise printed in Acta to see whether anything were to appear there which might form a motive for only publishing the first section in Crelle's journal, then one halts at the recursion formula (66) in §2, which constitutes one of the principal results of §2, and which is necessary for the proof of the main results appearing later in the treatise. This equation is not deduced from the calculation described there, unless one assumes a relation between some of the auxiliary terms.

To be able to explain this, I have used Abel's original form for presenting elliptic functions.[3] He puts the elliptic integral in the form:

$$\int_0^x \frac{dx}{\sqrt{1-c^2x^2 \,.\, 1+e^2x^2}};$$

if u represents this, then he puts

$$x = \varphi\,(u;\, c,\, e), \text{ or shorter } = \varphi\,(u).$$

The real and the purely imaginary periods are denoted 2ω and $2\varpi i$, respectively, where

$$\omega = 2\int_0^{\frac{1}{c}} \frac{d_x}{\sqrt{1-c^2x^2 \,.\, 1+e^2x^2}}, \quad \tilde{\omega} = 2\int_0^{\frac{1}{e}} \frac{d_x}{\sqrt{1+c^2x^2 \,.\, 1-e^2x^2}}.$$

Furthermore, if one sets

$$\alpha = \frac{2\omega}{2n+1}, \quad \beta = \frac{2\tilde{\omega}\, i}{2n+1}, \quad \delta = \cos\frac{2\pi}{2n+1} + i\sin\frac{2\pi}{2n+1},$$

then the auxiliary terms mentioned above, denoted c_k, are defined by the equation:

$$c_k = \frac{1}{2n+1}\sum_{0\ m}^{2n+1}\ \sum_{0\ \mu}^{2n+1} \delta^{mk}\,\varphi\left(m\,\alpha + \mu\,\beta\right)$$

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[2] 27th Volume of Acta Mathematica was displayed on the Chairman's table.

[3] In the lecture, the formulae were written on a carton.

and the equation connecting these, is

$$c_{k+1} \cdot c_{k-1} = \frac{c_k^2 - c_1^2}{1 + e^2 c^2 c_1^2 c_k^2}.$$

But no such formula is proven in §2. If one reads on, one finds (although only in §4) the equation (102) which in reality is a characterization of c_k as an elliptic function:

$$c_k = (-1)^n \varphi\left(nk\frac{\beta}{2}\right).$$

However, even here this is not put in connection with §2; it seems as if Abel is no longer thinking of section 2. By means of this expression for c_k, the equation connecting the c_k's, which we mentioned above, changes to an elementary elliptic formula, known from Abel's first treatises. The gap is remedied in that respect.

But then the equation (102) has not been proven *completely*, since Abel had omitted the rather intricate determination of two integers which originally appeared there – neither is it quite correct as it appears in print in Acta Mathematica. All-in-all, this is a weakness in the *editing* of the treatise of which Abel may indeed have become aware of after submission, and which may have made him dissatisfied with it.

Neither does the content of the treatise quite match the words in the introduction. However, his discovery of his addition theorem was not caused by Jacobi's Note, which one might be led to believe from the introduction. It is of course known that Abel discovered this prior to his journey abroad.

There is also another consideration that may have made Abel decide to have only the first section of "Second Mémoire" printed in Crelle. Once he had decided to write his last treatise on elliptic functions [Précis ···], which was to contain a selection of all his discoveries in this field, it was perhaps not a good idea to deprive the work of some of its novelty by reporting individual results in advance. It might have been of more interest to him that Précis would attract maximal attention because of the plans that were in motion to invite him to take up a professorship in Berlin, and for the favourable impression that he would like to make at the start of his appointment, if this came prior to the publication. The date for the printing of "Second Mémoire" was postponed to December 1828, namely to Number 4 of the 3rd Volume of Crelle's Journal, probably due to lack of space. Number 3 had been issued as early as September 21 and its content must have been determined far in advance of that.

I believe therefore that Abel *himself* decided to shorten this "Second Mémoire". When he had the first section printed, which gives the division equation's monodromy group, he gave his answer to Jacobi's short Note in a concise manner. I can well imagine that Crelle may have advised this, but not that he had arbitrarily made any changes to Abel's manuscript. The relationship between Abel and Crelle gives the impression of being such a good friendship that I would reject any thought that might cast a shadow over it.

The relationship of the second "Second Mémoire" to that published in Acta Mathematica is not clear. It is conceivable that Abel began work on it before he saw Jacobi's Note, and then put it aside. That is the most probable explanation. However, it is of course conceivable that he, for a time, wanted to rework the one that he had sent because he saw it would be a long time before it would appear.

I will move on to the letter from Abel to Crelle, which Prof Mittag-Leffler saved from oblivion. It is as he says, of great interest, because it provides an explanation of several things that were obscure before. It was of course known that Abel had been ill in the summer of 1828, and was only reasonably well again on September 22. However, the letter shows that the illness had left him in such a state of weakness that the doctor had advised him to refrain from all exertion – however, Abel did not follow the advice very conscientiously. That is, apparently after September he was working on editing his last treatise on elliptic functions, Précis. As excellent as this treatise is, and so rich in new discoveries, it bears traces of his impaired health. The introduction, in particular, which enumerates a series of the most important results, contains things that puzzled my colleague Sophus Lie and myself, and which has also surprised other mathematicians. In the introduction, Abel certainly wrote from memory, without the full scope of his own discoveries always being quite clear to him and with lapses in his memory. I ought here to mention a couple of the most important examples.

He says of the “modular equations” that each of their roots may be expressed rationally by two of them. In fact, of course, that is not the case – one sees that now almost immediately from Galois theory. There may be a failure of memory here, a confusion. It is the period division equations that have this property. Both classes of equations are so closely related to one another that momentary confusion is conceivable. There should hardly be any doubt about the correctness of this explanation. Mr Mittag-Leffler has also found it probable. An outstanding German mathematician, Kronecker, said about this passage: “wie hat uns das Mühe gemacht: wir wollten es beweisen und es war falsch!”

A passage that is basically more important is where he speaks of the singular moduli – those that give rise to complex multiplication. In a previous treatise, Abel had said with certainty that these are roots in algebraic equations that can be solved by root extractions. In Précis, his last work, he expresses himself in a more reserved manner; he says that it *seems* to be possible to solve these equations by root extractions, that this is at least the case when the absolute value of the quotient of the periods is a rational number. But with this remark, the scope of the theorem is limited greatly, much too much in fact, for Abel demonstrably knew other situations in which this was true. If his words are perceived as a retraction of the previous statement, then the credit for discovering this theorem should not go to Abel but to Kronecker. I believe that it is now possible to understand Abel’s words to mean, in all probability, that they do not contain any retraction. It would not of course have mattered if the paper contained more than what had been stated in the introduction. It was the opposite he did not want to expose himself to. It is poignant to see how this great researcher’s ardour does not allow him, even temporarily, to stop work in spite of the fact that his powers were failing at that moment.

Previously, I have always believed that the reason why Abel had not written a comprehensive work dealing with algebraic equations was the competition with Jacobi which consumed so much time and energy. Abel’s letter shows that, to a certain extent, I have been wrong in this. It was, as the letter informs us, agreed between Crelle and himself that he would first write about the theory of equations and later about elliptic functions. But now Abel is afraid that the theory of equation will be more of a strain on his health than the other topic, and suggests starting with elliptic functions instead.

The question is what would the content of such a great work on equations have been like? What Abel previously mentioned as the subject matter of his work in this field, can be attributed to two classes. The first class contains applications of his theory of abelian equations to elliptic functions, in particular, the solution of the division equation and the modular equation for singular moduli, and of the equation for the singular moduli themselves. The second class includes the general theory of solutions of algebraic equations by radial expressions. The questions raised by the first class would be most naturally subsumed under the theory of elliptic functions. Essentially it is a matter of having to prove that the equation relevant is an abelian equation, and that would of course have to be shown on the basis of function theory. Therefore, it is only the second class that should probably have been the subject of the presumptive treatise on the theory of equations.

Evaluated on the basis of where Abel entered it in the book where he wrote his mathematical notes, the preliminary draft to a comprehensive work on the theory of equations – published in both editions of Abel’s Collected Works – was written in the second half of 1828. The letter to Crelle shows that he hardly worked on it at all after the date of the letter, September 25, since he requested to be released from this obligation until further notice. And yet on July 29, he was still occupied with the second of the two treatises, which he had published in Astronomische Nachrichten. It is therefore now known that Abel’s work on this draft took place between these two dates, that it was therefore definitely written in part during his illness and was discontinued because of it. This is an important piece of information for the understanding of the work; the fact that he left it incomplete is not because he did not believe himself capable of bringing it to its completion.

Abel undoubtedly observed that there was a weak point in the draft, in one of the theorems on which the theory was based. This defect was not of any greater significance, and that all objections would be put to rest with a more exhaustive treatment. I believe I have shown this in the notes to the second edition and to have given a satisfactory interpretation of those parts of the draft that have not been fully edited, so that the results are complete proofs for the theorems that Abel states for equations of degree a prime, a prime power, or the product of pairmise different prime numbers, respectively. After that, to

Abel, the difficulties should not seem so great. But just after his illness, he perhaps saw them as being greater than they really were. In any case, the awareness that he would have to revise what he had written brought about the wish in him to have more time and more energy for carrying out the revision. However it should be remembered that, in the draft, where it has been completed, Abel goes beyond these theorems; for example, where it relates to equations of degree a prime number. If he also had in mind to treat the case when the degree is a prime power, the difficulties would in truth be great. One merely has to compare Jordan's Traité. But we do not know anything of that.

Abel's health went rapidly downhill in the year 1828. His journeys appear not to have agreed with him at all. He was ill for some days immediately after his arrival at Froland in the summer. He became ill once more on his return to Christiania, but seriously this time. But he still did not spare himself from arduous work. In the year 1828 he wrote over a third of everything that is contained in the second edition of his Collected Works, including the last treatise on elliptic functions, Précis, which he wrote after the month of September, when he had been warned by his doctor against overexerting himself. There is no doubt that all of this work contributed to shortening his life. When he came to Froland at Christmas he took to bed with his final illness. The pneumonia probably lifted, but was followed by a state of severe weakness. The tuberculosis returned and it killed him. It is hard to think that this person, who had so much to achieve, should leave us so early. But even if he had avoided the fatal pneumonia it is not certain that to take up a professorship in Berlin – which came about, it seemed, after much opposition, but, at the same time, also with great expectations – would have been easy and pleasant for this man with such severe health problems.

But there is *one* redeeming feature however in Abel's tragic history – he passed away with the awareness that his services to science had been acknowledged in full.