



# Yakov G. Sinai, Abel Prize Laureate 2014

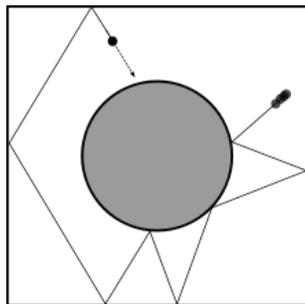


## DYNAMICAL BILLIARD

**A dynamical billiard is an idealization of the game of billiard, but where the table can have shapes other than the rectangular and even be multidimensional. We use only one billiard ball, and the billiard may even have regions where the ball is kept out.**

Formally, a dynamical billiard is a dynamical system where a massless and point shaped particle moves inside a bounded region. The particle is reflected by specular reflections at the boundary, without loss of speed. In between two reflections the particle moves rectilinear at constant speed. Remember that a specular reflection is characterized by the law of reflection, the angle of incidence equals the angle of reflection.

An example of a dynamical billiard is the so-called Sinai's billiard. The table of the Sinai billiard is a square with a disk removed from its center; the table is flat, having no curvature



Sinai's billiard

(Source: Georg Stamatou, Wikipedia)

The billiard ball is reflected alternately from the outer and the inner boundary.

Sinai's billiard arises from studying the model of the behavior of molecules in a so-

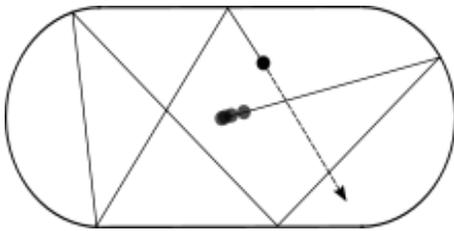
called ideal gas. In this model we consider the gas as numerous tiny balls (gas molecules) bouncing inside a square, reflecting off the boundaries of the square and off each other. Sinai's billiard provides a simplified, but rather good illustration of this model.

The billiard was introduced by Yakov G. Sinai as an example of an interacting Hamiltonian system that displays physical thermodynamic properties: all of its possible trajectories are **ergodic**, and it has positive Lyapunov exponents. Thus the system shows chaotic behavior. As a model of a classical gas, the Sinai billiard is sometimes called the Lorentz gas. Sinai's great achievement with this model was to show that the behavior of the gas molecules follows the trajectories of the Hadamard dynamical system, as described by Hadamard in 1898, in the first paper that studied mathematical chaos systematically.

A dynamical billiard doesn't have to be planar. In case of non-zero curvature rectilinear motion is replaced by motion along geodesics, i.e. curves which give the shortest path between points in the billiard. The motion of the ball is geodesic of constant speed, thus the trajectories are completely described by the reflections at the boundary. The system is deterministic, thus if we know the position and the angle of one reflection, the whole trajectory can be determined. The map that takes one state to the next is called the billiard transformation. The billiard transformation determines the dynamical system.

In the ordinary rectangular billiard we observe no chaotic behavior. A small change in the initial data will induce significant devia-

tion in the long run, but the deviation will be a linear function of time. Chaotic behavior is characterized by exponential growth in the deviation. For Sinai's billiard chaotic behavior is observed. For a long time it was assumed that the reason for the exponential deviation of trajectories that are close to each other was the concave shape of the inner boundary. It was also believed that a concave shape was necessary to obtain the chaotic behavior, just like a concave lens spreads the light. But in 1974 Leonid Bunimovich proved that a billiard table shaped like a stadium, where two opposing sides are replaced by semicircles, produces chaotic behavior, in spite of the fact that this billiard is completely convex.



Bunimovich billiard

(Source: Georg Stamatiou, Wikipedia)

### An example

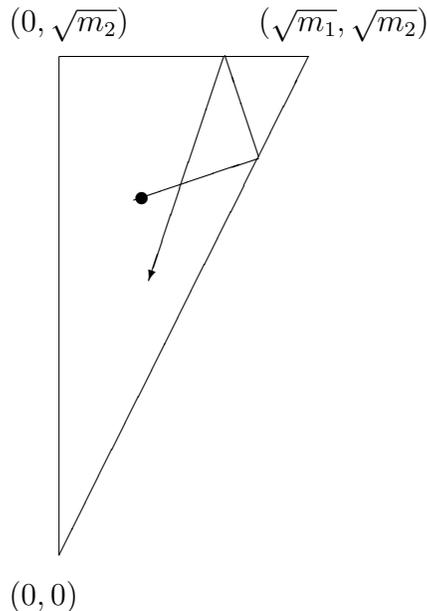
Consider the following example of a billiard. The physical model consists of two molecules, moving in a one-dimensional interval  $[0, 1]$ . When a molecule hits an endpoint, it is reflected elastically, i.e. the velocity is the same, in opposite direction. Collisions between the two molecules are elastic as well, conserving momentum and energy. Let the mass of the two molecules be  $m_1$  and  $m_2$ . Suppose that the velocities of the two

molecules are  $v_1$  and  $v_2$  before the collision and  $w_1$  and  $w_2$  after the collision. Thus we have the two equations

$$m_1v_1 + m_2v_2 = m_1w_1 + m_2w_2$$

$$\frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 = \frac{1}{2}m_1w_1^2 + \frac{1}{2}m_2w_2^2$$

The positions of the two molecules are given by the coordinates,  $x_1$  and  $x_2$ , written as a pair  $(x_1, x_2)$ . This pair describes a state of the system. The state space parametrizes all possible states. In this example the two molecules are placed on the interval  $[0, 1]$ , thus  $0 \leq x_1 \leq 1$  and  $0 \leq x_2 \leq 1$ . We also assume that the two molecules are ordered, with molecule 1 to the left of molecule 2, i.e.  $x_1 \leq x_2$ . We form a triangle shaped billiard table, with vertices  $(0, 0)$ ,  $(0, \sqrt{m_2})$  and  $(\sqrt{m_1}, \sqrt{m_2})$ .



A point  $(x, y)$  refers to molecule no. 1 placed at  $x_1 = \frac{x}{\sqrt{m_1}}$  and molecule no. 2 placed at  $x_2 = \frac{y}{\sqrt{m_2}}$ . where the molecules

collide, we have  $x_1 = x_2$ , which means that  $(x, y)$  is placed at the hypotenuse of the billiard boundary.

Conservation of momentum and energy gives

$$w_1 = \frac{2m_2v_2 + (m_1 - m_2)v_1}{m_1 + m_2}$$

$$w_2 = \frac{2m_1v_1 + (m_2 - m_1)v_2}{m_1 + m_2}$$

and thus  $v_2 - v_1 = w_1 - w_2$ . Reflection along the hypotenuse of the billiard is determined by

$$(\sqrt{m_1}v_1, \sqrt{m_2}v_2) \cdot (-\sqrt{m_2}, \sqrt{m_1})$$

$$= -(\sqrt{m_1}w_1, \sqrt{m_2}w_2) \cdot (-\sqrt{m_2}, \sqrt{m_1})$$

or  $-v_1 + v_2 = w_1 - w_2$ . Thus, the triangle billiard gives a complete description of the dynamics of the one-dimensional given gas model.