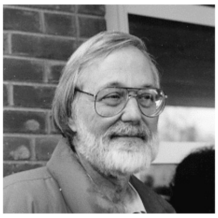


**Abel Prize Laureate 2011**

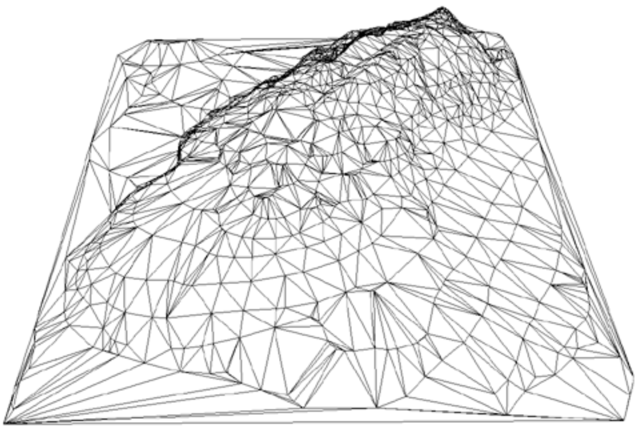
John Willard Milnor

# *Die Hauptvermutung der kombinatorischen Topologie (Steiniz, Tietze; 1908)*

**Die Hauptvermutung (The main Conjecture) of combinatorial topology (now: algebraic topology) was published in 1908 by the German mathematician Ernst Steiniz and the Austrian mathematician Heinrich Tietze. The conjecture states that given two triangulations of the same space, there is always possible to find a common refinement. The conjecture was proved in dimension 2 by Tibor Radó in the 1920s and in dimension 3 by Edwin E. Moise in the 1950s. The conjecture was disproved in dimension greater or equal to 6 by John Milnor in 1961.**

**Triangulation**

"Norges Geografiske Oppmåling" (NGO) was established in 1773 by the military officer Heinrich Wilhelm von Huth with the purpose of measuring Norway in order to draw precise and useful maps. Six years later they started the rather elaborate triangulation task. When triangulating a piece of land you have to pick reference points and compute their coordinates relative to nearby points. Next you chose a bunch of connecting lines, edges, between the reference points in order to obtain a tiangular web. The choices of reference points and edges are done in order to obtain triangles where the curvature of the interior landscape is neglectable. Thus, if the landscape is hilly, the reference points have to be chosen rather dense, while flat farmland doesn´t need many points. In this way it is possible to give a rather accurate description of the whole landscape. The recipe of triangulation can be used for arbitrary surfaces. By the process we obtain a piecewise linearisation (PL) of the surface. The advantage of the PL model is its simplicity, combinatorial and computational. At the same time the essential properties of the surface are maintained.



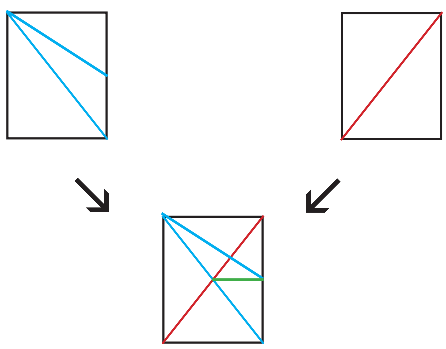
*Triangulering av et fjellandskap*

**Refinements of triangulations**

In the *Hauptvermutung* the concept of refinement of a triangulation is crucial. A refinement of a triangulation is obtained when introducing additional reference points and edges, refining the existing triangular web. The refinement is not unique. In fact there are infinitely many choices of reference points and a finite number of possible choices of edges to create a triangular web. The Hauptvermutung tells us that for any two original triangulations of a space, we can always find a common refinement.

For plane domains the refinement procedure can be easily described. The two traingulations are overlayed, with all reference points and edges of both triangulations as the new skeleton. It is possible that two edges meet in a non-reference point. We then introduce a new reference point in the intersection. It is also possible that this configuration includes polygons with more than three edges. In that case we introduce enough new points and edges to maintain the triangular structure.

It is important to notice that the Hauptvermutung concerns any pair of triangulations of a given space. Thus, to disprove the conjecture, one counterexample is all we need. In 1961 Milnor gave such a counterexample.



*Two triangulations of a rectangle and their common refinement*

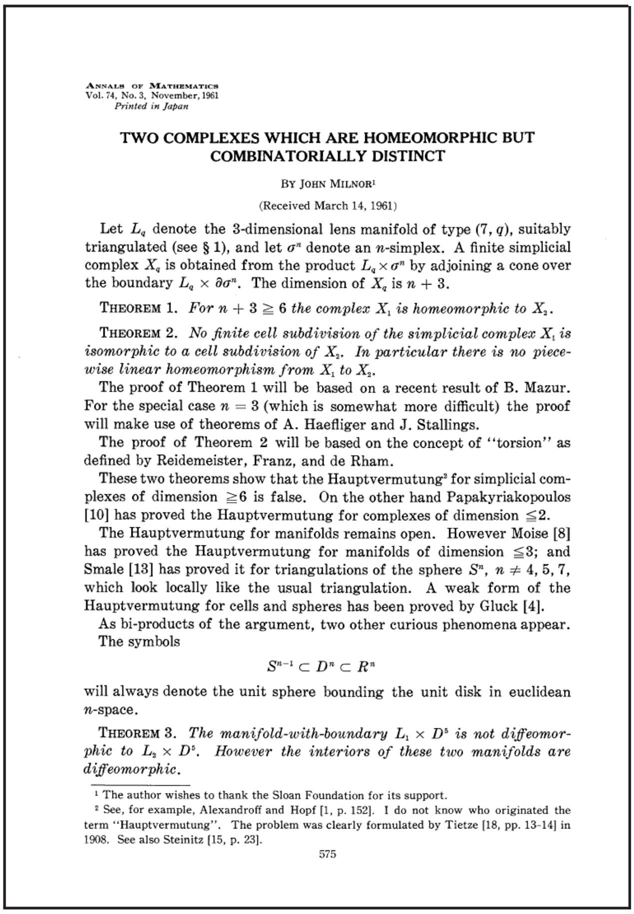
### Milnors teorem

Let $L_q$ denote the 3-dimensional lens manifold of type $(7, q)$, suitably triangulated, and let $\Delta^n$ denote an n-simplex. A finite simplicial complex $X_q$ is obtained from the product $L_q \times \Delta^n$ by adjoining a cone over the boundary $L_q \times \partial\Delta^n$. The dimension of $X_q$ is $n+3$.

***Theorem 1***. For $n+3 \geq 6$ the complex $X_1$ is homeomorphic to $X_2$.

***Theorem 2***. No finite cell subdivisions of the simplicial complex $X_1$ is isomorphic to a cell subdivision of $X_2$.

The theorem gives a counterexample to the Hauptvermutung of combinatorial topology.



ANNALS OF MATHEMATICS
Vol. 74, No. 3, November, 1961
*Printed in Japan*

TWO COMPLEXES WHICH ARE HOMEOMORPHIC BUT COMBINATORIALLY DISTINCT

BY JOHN MILNOR[1]

(Received March 14, 1961)

Let $L_q$ denote the 3-dimensional lens manifold of type $(7, q)$, suitably triangulated (see § 1), and let $\sigma^n$ denote an $n$-simplex. A finite simplicial complex $X_q$ is obtained from the product $L_q \times \sigma^n$ by adjoining a cone over the boundary $L_q \times \partial\sigma^n$. The dimension of $X_q$ is $n + 3$.

THEOREM 1. *For $n + 3 \geqq 6$ the complex $X_1$ is homeomorphic to $X_2$.*

THEOREM 2. *No finite cell subdivision of the simplicial complex $X_1$ is isomorphic to a cell subdivision of $X_2$. In particular there is no piecewise linear homeomorphism from $X_1$ to $X_2$.*

The proof of Theorem 1 will be based on a recent result of B. Mazur. For the special case $n = 3$ (which is somewhat more difficult) the proof will make use of theorems of A. Haefliger and J. Stallings.

The proof of Theorem 2 will be based on the concept of "torsion" as defined by Reidemeister, Franz, and de Rham.

These two theorems show that the Hauptvermutung[2] for simplicial complexes of dimension $\geqq 6$ is false. On the other hand Papakyriakopoulos [10] has proved the Hauptvermutung for complexes of dimension $\leqq 2$.

The Hauptvermutung for manifolds remains open. However Moise [8] has proved the Hauptvermutung for manifolds of dimension $\leqq 3$; and Smale [13] has proved it for triangulations of the sphere $S^n$, $n \neq 4, 5, 7$, which look locally like the usual triangulation. A weak form of the Hauptvermutung for cells and spheres has been proved by Gluck [4].

As bi-products of the argument, two other curious phenomena appear. The symbols

$$S^{n-1} \subset D^n \subset R^n$$

will always denote the unit sphere bounding the unit disk in euclidean $n$-space.

THEOREM 3. *The manifold-with-boundary $L_1 \times D^5$ is not diffeomorphic to $L_2 \times D^5$. However the interiors of these two manifolds are diffeomorphic.*

[1] The author wishes to thank the Sloan Foundation for its support.

[2] See, for example, Alexandroff and Hopf [1, p. 152]. I do not know who originated the term "Hauptvermutung". The problem was clearly formulated by Tietze [18, pp. 13-14] in 1908. See also Steinitz [15, p. 23].