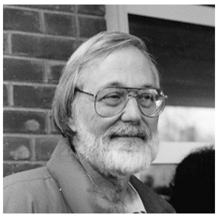


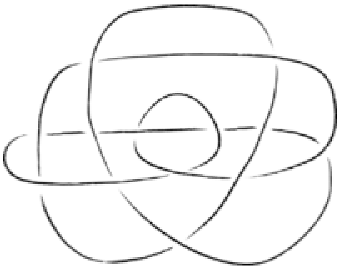
Abel Prize Laureate 2011
John Willard Milnor

J.W.Milnor: On the total Curvature of Knots

Annals of Mathematics, Vol 52, no 2 (1950)

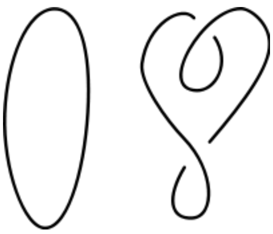
In October 1949 John Milnor got the message that his first paper was accepted for publication in an international journal. At that time he was only 18 years old. The paper was about "knot geometry". The German mathematician Werner Fenchel, at the University of Copenhagen, showed in 1929 that the total curvature of a closed space curve always exceeds 2π. The result was generalised to arbitrary dimensions by the polish mathematician Karol Borsuk in 1949. The theorem of Milnor combines Fenchel-Borsuk and knot theory, and states that for a non-trivial knot, the total curvature exceeds 4π, i.e. at least two rotations. The theorem was proven indepently, but almost simultanously, by the hungarian mathematician István Fáry. This is the reason for the name Fáry-Milnor´s theorem.

Knot theory
Knot theory is a subfield of mathematics aiming to describe all knots. The definition of a knot is a closed curve in space, i.e no open ends. We can illustrate knots by planar sketches, where each crossing has a prescribed way of telling which branch is above and which is beneath.



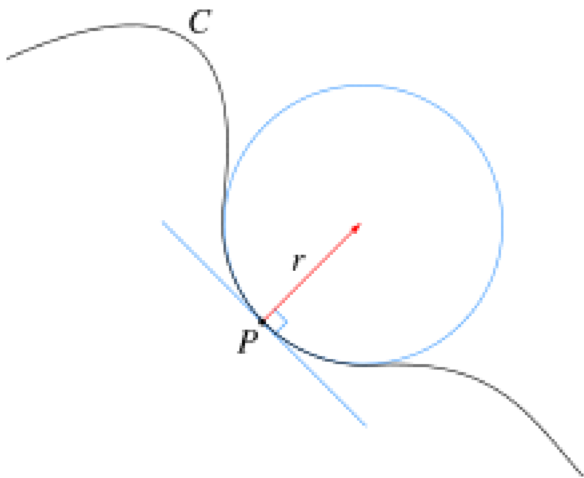
An example of a knot

Two knots are said to be equivalent if we can transform one into the other by pulling and pushing branches of the rope, but not cutting or gluing.

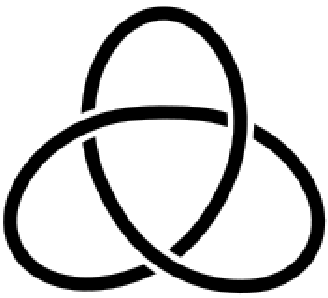


The simplest of all knots is the circle. It is often called an unknot since it is mathematically a knot, but normally not viewed as a knot. The illustration gives two versions of an unknot, the left one is obviously an unknot, and the right one can be transformed into the circle in an admissible way.

Curvature of space curves
Milnor´s first paper is about curvature of knots. The curvature of a curve is a function on the curve, where we to each point of the curve give a number, the curvature of the curve in that point. A straight line has curvature 0 in all points, and



a circle has constant curvature equal to 1 divided by the radius. Thus a smaller circle has greater curvature and vice versa. The total curvature is obtained by adding the curvature in all points of the curve. For a circle, of constant curvature, the total curvature is the circumference multiplied by the constant curvature,



$(1/R)* 2\pi R= 2\pi$

This fits perfect with the result of Fenchel from 1929, which claims that the total curvature for a closed curve is at least 2π, and equality holds for plane, convex curves.

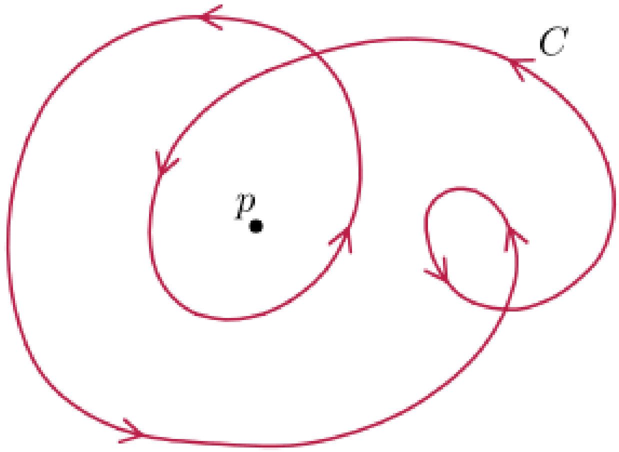
Fáry-Milnor´s theorem

Milnor´s result from 1949 is known as the Fáry-Milnor theorem. The reason for the double name is that the Hungarian mathematician István Fáry independently and at the same time also proved the result.

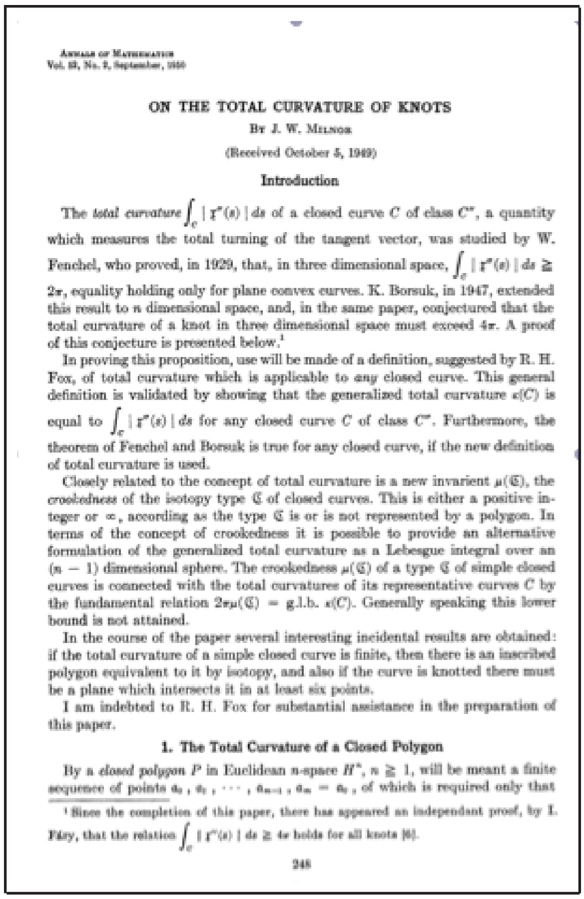
The Fáry-Milnor theorem claims that if a knot is not an unknot, the total curvature has to exceed 4π, i.e. the rope must be turned at least two times around to produce a non-trivial knot. The simplest non-trivial knot is the trefoil knot (see illustration above). By inspection it is easy to accept that this knot has total curvature at least 4π. Disregarding the parts of the curve where it crosses itself, the plane projection of the knot will have total curvature 4π. In the crossing, where one branch has to be lifted, there has to be some curvature in the direction out of the paper. Adding up we get a bit more than 4π.

Notice also that the Fáry-Milnor theorem only gives an implication one way; if the knot is an unknot, the curvature exceeds 4π. But the opposite statement is not true. As illustrated in the next figure, there are unknots of total curvature much greater than 4π.

The proof of Milnor´s theorem for curvature of knots does not involve very hard mathematics, but



it is rather elegant. The mathematics community was a bit surprised that a youth of age 18 could prove such a theorem. Also, the paper showed much maturity. The mathematics community had discovered a great mathematical talent.



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ON THE TOTAL CURVATURE OF KNOTS

BY J. W. MILNOR

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Introduction

The *total curvature* $\int_C |\mathfrak{r}''(s)|\,ds$ of a closed curve C of class C'', a quantity which measures the total turning of the tangent vector, was studied by W. Fenchel, who proved, in 1929, that, in three dimensional space, $\int_C |\mathfrak{r}''(s)|\,ds \geqq 2\pi$, equality holding only for plane convex curves. K. Borsuk, in 1947, extended this result to n dimensional space, and, in the same paper, conjectured that the total curvature of a knot in three dimensional space must exceed 4π. A proof of this conjecture is presented below.[1]

In proving this proposition, use will be made of a definition, suggested by R. H. Fox, of total curvature which is applicable to *any* closed curve. This general definition is validated by showing that the generalized total curvature $\kappa(C)$ is equal to $\int_C |\mathfrak{r}''(s)|\,ds$ for any closed curve C of class C''. Furthermore, the theorem of Fenchel and Borsuk is true for any closed curve, if the new definition of total curvature is used.

Closely related to the concept of total curvature is a new invarient $\mu(\mathfrak{C})$, the *crookedness* of the isotopy type $\mathfrak{C}$ of closed curves. This is either a positive integer or ∞, according as the type $\mathfrak{C}$ is or is not represented by a polygon. In terms of the concept of crookedness it is possible to provide an alternative formulation of the generalized total curvature as a Lebesgue integral over an $(n-1)$ dimensional sphere. The crookedness $\mu(\mathfrak{C})$ of a type $\mathfrak{C}$ of simple closed curves is connected with the total curvatures of its representative curves C by the fundamental relation $2\pi\mu(\mathfrak{C}) = \text{g.l.b. } \kappa(C)$. Generally speaking this lower bound is not attained.

In the course of the paper several interesting incidental results are obtained: if the total curvature of a simple closed curve is finite, then there is an inscribed polygon equivalent to it by isotopy, and also if the curve is knotted there must be a plane which intersects it in at least six points.

I am indebted to R. H. Fox for substantial assistance in the preparation of this paper.

1. The Total Curvature of a Closed Polygon

By a *closed polygon* P in Euclidean n-space E^n, $n \geqq 1$, will be meant a finite sequence of points $a_0, a_1, \cdots, a_{m-1}, a_m = a_0$, of which is required only that

[1] Since the completion of this paper, there has appeared an independant proof, by I. Fáry, that the relation $\int_C |\mathfrak{r}''(s)|\,ds \geqq 4\pi$ holds for all knots [6].

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